

Learning Active Basis Models by EM-Type Algorithms

Zhangzhang Si , Haifeng Gong , Song-Chun Zhu and Ying Nian Wu

Abstract. EM algorithm is a convenient tool for maximum likelihood model fitting when the data are incomplete or when there are latent variables or hidden states. In this review article we explain that EM algorithm is a natural computational scheme for learning image templates of object categories where the learning is not fully supervised. We represent an image template by an active basis model, which is a linear composition of a selected set of localized, elongated and oriented wavelet elements that are allowed to slightly perturb their locations and orientations to account for the deformations of object shapes. The model can be easily learned when the objects in the training images are of the same pose, and appear at the same location and scale. This is often called supervised learning. In the situation where the objects may appear at different unknown locations, orientations and scales in the training images, we have to incorporate the unknown locations, orientations and scales as latent variables into the image generation process, and learn the template by EM-type algorithms. The E-step imputes the unknown locations, orientations and scales based on the currently learned template. This step can be considered self-supervision, which involves using the current template to recognize the objects in the training images. The M-step then relearns the template based on the imputed locations, orientations and scales, and this is essentially the same as supervised learning. So the EM learning process iterates between recognition and supervised learning. We illustrate this scheme by several experiments.

Key words and phrases: Generative models, object recognition, wavelet sparse coding.

Zhangzhang Si is Ph.D. Student, Department of Statistics, University of California, Los Angeles, USA e-mail: zzsi@stat.ucla.edu. Haifeng Gong is Postdoctoral Researcher, Department of Statistics, University of California, Los Angeles, USA and Lotus Hill Research Institute, Ezhou, China e-mail: hfgong@stat.ucla.edu. Song-Chun Zhu is Professor, Department of Statistics, University of California, Los Angeles, USA and Lotus Hill Research Institute, Ezhou, China e-mail: sczhu@stat.ucla.edu. Ying Nian Wu is Professor, Department of Statistics, University of California, Los Angeles, USA e-mail: ywu@stat.ucla.edu.

1. INTRODUCTION: EM LEARNING SCHEME

The EM algorithm [7] and its variations [14] have been widely used for maximum likelihood estimation of statistical models when the data are incompletely observed or when there are latent variables or hidden states. This algorithm is an iterative computational scheme, where the E-step imputes the missing data or the latent variables given the currently

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estimated model, and the M-step re-estimates the model given the imputed missing data or latent variables. Besides its simplicity and stability, a key feature that distinguishes the EM algorithm from other numerical methods is its interpretability: both the E-step and the M-step readily admit natural interpretations in a variety of contexts. This makes the EM algorithm rich, meaningful and inspiring.

In this review article we shall focus on one important context where the EM algorithm is useful and meaningful, that is, learning patterns from signals in the settings that are not fully supervised. In this context, the E-step can be interpreted as carrying out the recognition task using the currently learned model of the pattern. The M-step can be interpreted as relearning the pattern in the supervised setting, which can often be easily accomplished.

This EM learning scheme has been used in both speech and vision. In speech recognition, the training of the hidden Markov model [16] involves the imputation of the hidden states in the E-step by the forward and backward algorithms. The M-step computes the transition and emission frequencies. In computer vision, we want to learn models for different categories of objects, such as horses, birds, bikes, etc. The learning is often easy when the objects in the training images are aligned, in the sense that the objects appear at the same pose, same location and same scale in the training images, which are defined on a common image lattice that is the bounding box of the objects. This is often called supervised learning. However, it is often the case that the objects appear at different unknown locations, orientations and scales in the training images. In such a situation, we have to incorporate the unknown locations, orientations and scales as latent variables in the image generation process, and use the EM algorithm to learn the model for the objects. In the EM learning process, the E-step imputes the unknown location, orientation and scale of the object in each training image, based on the currently learned model. This step uses the current model to recognize the object in each training image, that is, where it is, at what orientation and scale. The imputation of the latent variables enables us to align the training images, so that the objects appear at the same location, orientation and scale. The M-step then relearns the model from the aligned images by carrying out supervised learning. So the EM learning process iterates between recognition and supervised learning. Recognition is the goal of learning the model, and

it serves as the self-supervision step of the learning process. The EM algorithm has been used by Fergus, Perona and Zisserman [9] in training the constellation model for objects.

In this article we shall illustrate EM learning or EM-like learning by training an active basis model [22, 23] that we have recently developed for deformable templates [1, 24] of object shapes. In this model, a template is represented by a linear composition of a set of localized, elongated and oriented wavelet elements at selected locations, scales and orientations, and these wavelet elements are allowed to slightly perturb their locations and orientations to account for the shape deformations of the objects. In the supervised setting, the active basis model can be learned by a shared sketch algorithm, which selects the wavelet elements sequentially. When a wavelet element is selected, it is shared by all the training images, in the sense that a perturbed version of this element seeks to sketch a local edge segment in each training image. In the situations where learning is not fully supervised, the learning of the active basis model can be accomplished by the EM-type algorithms. The E-step recognizes the object in each training image by matching the image with the currently learned active basis template. This enables us to align the images. The M-step then relearns the template from the aligned images by the shared sketch algorithm.

We would like to point out that the EM algorithm for learning the active basis model is different than the traditional EM algorithm, where the model structure is fixed and only the parameters need to be estimated. In our implementation of the EM algorithm, the M-step needs to select the wavelet elements in addition to estimating the parameters associated with the selected elements. Both the selection of the elements and the estimation of the parameters are accomplished by maximizing or increasing the complete-data log-likelihood. So the EM algorithm is used for estimating both the model structure and the associated parameters.

Readers who wish to learn more about the active basis model are referred to our recent paper [23], which is written for the computer vision community. Compared to that paper, this review paper is written for the statistical community. In this paper we introduce the active basis model from an algorithmic perspective, starting from the familiar problem of variable selection in linear regression. This paper

also provides more details about the EM-type algorithms than [23]. We wish to convey to the statistical audience that the problem of vision in general and object recognition in particular is essentially a statistical problem. We even hope that this article may attract some statisticians to work on this interesting but challenging problem.

Section 2 introduces the active basis model for representing deformable templates, and describes the shared sketch algorithm for supervised learning. Section 3 presents the EM algorithm for learning the active basis model in the settings that are not fully supervised. Section 4 concludes with a discussion.

2. ACTIVE BASIS MODEL: AN ALGORITHMIC TOUR

The active basis model is a natural generalization of the wavelet regression model. In this section we first explain the background and motivation for the active basis model. Then we work through a series of variable selection algorithms for wavelet regression, where the active basis model emerges naturally.

2.1 From Wavelet Regression to Active Basis

2.1.1 $p > n$ regression and variable selection Wavelets have proven to be immensely useful for signal analysis and representation [8]. Various dictionaries of wavelets have been designed for different types of signals or function spaces [4, 19]. Two key factors underlying the successes of wavelets are the sparsity of the representation and the efficiency of the analysis. Specifically, a signal can typically be represented by a linear superposition of a small number of wavelet elements selected from an appropriate dictionary. The selection can be accomplished by efficient algorithms such as matching pursuit [13] and basis pursuit [5].

From a linear regression perspective, a signal can be considered a response variable, and the wavelet elements in the dictionary can be considered the predictor variables or regressors. The number of elements in a dictionary can often be much greater than the dimensionality of the signal, so this is the so-called “ $p > n$ ” problem. The selection of the wavelet elements is the variable selection problem in linear regression. The matching pursuit algorithm [13] is the forward selection method, and the basis pursuit [5] is the lasso method [21].

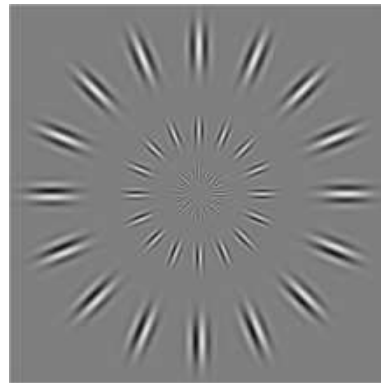


FIG. 1. A collection of Gabor wavelets at different locations, orientations and scales. Each Gabor wavelet element is a sine or cosine wave multiplied by an elongated and oriented Gaussian function. The wave propagates along the shorter axis of the Gaussian function.

2.1.2 Gabor wavelets and simple V1 cells Interestingly, wavelet sparse coding also appears to be employed by the biological visual system for representing natural images. By assuming the sparsity of the linear representation, Olshausen and Field [15] were able to learn from natural images a dictionary of localized, elongated, and oriented basis functions that resemble the Gabor wavelets. Similar wavelets were also obtained by independent component analysis of natural images [2]. From a linear regression perspective, Olshausen and Field essentially asked the following question: Given a sample of response vectors (i.e., natural images), can we find a dictionary of predictor vectors or regressors (i.e., basis functions or basis elements), so that each response vector can be represented as a linear combination of a small number of regressors selected from the dictionary? Of course, for different response vectors, different sets of regressors may be selected from the dictionary.

Figure 1 displays a collection of Gabor wavelet elements at different locations, orientations and scales. These are sine and cosine waves multiplied by elongated and oriented Gaussian functions, where the waves propagate along the shorter axes of the Gaussian functions. Such Gabor wavelets have been proposed as mathematical models for the receptive fields of the simple cells of the primary visual cortex or V1 [6].

The dictionary of all the Gabor wavelet elements can be very large, because at each pixel of the image domain, there can be many Gabor wavelet elements tuned to different scales and orientations. According to Olshausen and Field [15], the biological vi-

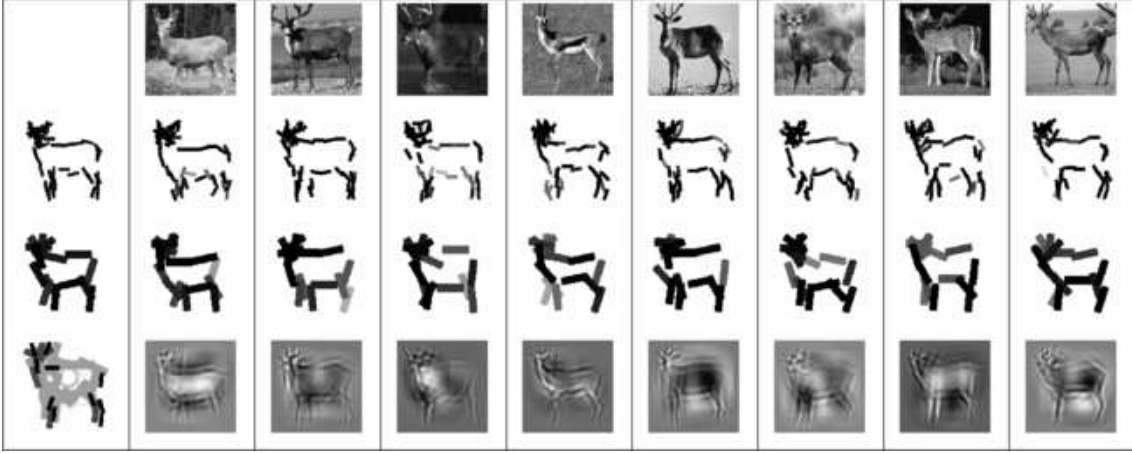


FIG. 2. *Active basis templates. Each Gabor wavelet element is illustrated by a bar of the same length and at the same location and orientation as the corresponding element. The first row displays the training images. The second row displays the templates composed of 50 Gabor wavelet elements at a fixed scale, where the first template is the common deformable template, and the other templates are deformed templates for coding the corresponding images. The third row displays the templates composed of 15 Gabor wavelet elements at a scale about twice as large as those in the second row. In the last row, the template is composed of wavelet elements at multiple scales, where larger Gabor elements are illustrated by bars of lighter shades. The rest of the images are reconstructed by linear superpositions of the wavelet elements of the deformed templates.*

sual system represents a natural image by a linear superposition of a small number of Gabor wavelet elements selected from such a dictionary.

2.1.3 From generic classes to specific categories

Wavelets are designed for generic function classes or learned from generic ensembles such as natural images, under the generic principle of sparsity. While such generality offers enormous scope for the applicability of wavelets, sparsity alone is clearly inadequate for modeling specific patterns. Recently, we have developed an active basis model for images of various object classes [22, 23]. The model is a natural consequence of seeking a common wavelet representation simultaneously for multiple training images from the same object category.

The active basis can be learned by the shared sketch algorithm that we have recently developed [22, 23]. This algorithm can be considered a paralleled version of the matching pursuit algorithm [13]. It can also be considered a modification of the projection pursuit algorithm [10]. The algorithm selects the wavelet elements sequentially from the dictionary. Each time when an element is selected, it is shared by all the training images in the sense that a perturbed version of this element is included in the linear representation of each image. Figure 3 illustrates the shared sketch process for obtaining the templates displayed in the second and third rows of Figure 2.

Figure 2 illustrates the basic idea. In the first row there are 8 images of deer. The images are of the same size of 122×120 pixels. The deer appear at the same location, scale and pose in these images. For these very similar images, we want to seek a common wavelet representation, instead of coding each image individually. Specifically, we want these images to be represented by similar sets of wavelet elements, with similar coefficients. We can achieve this by selecting a common set of wavelet elements, while allowing these wavelet elements to locally perturb their locations and orientations before they are linearly combined to code each individual image. The perturbations are introduced to account for shape deformations in the deer. The linear basis formed by such perturbable wavelet elements is called an active basis.

This is illustrated by the second and third rows of Figure 2. In each row the first plot displays the common set of Gabor wavelet elements selected from a dictionary. The dictionary consists of Gabor wavelets at all the locations and orientations, but at a fixed scale. Each Gabor wavelet element is symbolically illustrated by a bar at the same location and orientation and with the same length as the corresponding Gabor wavelet. So the active basis formed by the selected Gabor wavelet elements can be interpreted as a template, as if each element is a stroke for sketching the template. The templates in the second and third rows are learned using dictionaries of

$n = 5$	$n = 10$	$n = 20$	$n = 30$	$n = 40$	$n = 50$
$n = 1$	$n = 2$	$n = 3$	$n = 5$	$n = 10$	$n = 15$

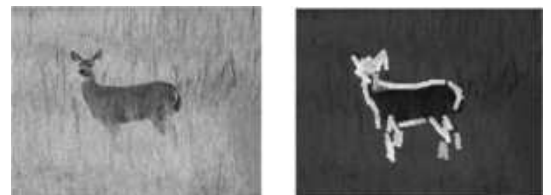
FIG. 3. *Shared sketch process for learning the active basis templates at two different scales.*

Gabor wavelets at two different scales, with the scale of the third row about twice as large as the scale of the second row. The number of Gabor wavelet elements of the template in the second row is 50, while the number of elements of the template in the third row is 15. Currently, we treat this number as a tuning parameter, although they can be determined in a more principled way.

Within each of the second and third rows, and for each training image, we plot the Gabor wavelet elements that are actually used to represent the corresponding image. These elements are perturbed versions of the corresponding elements in the first column. So the templates in the first column are deformable templates, and the templates in the remaining columns are deformed templates. Thus, the goal of seeking a common wavelet representation for images from the same object category leads us to formulate the active basis, which is a deformable template for the images from the object category.

In the last row of Figure 2, the common template is learned by selecting from a dictionary that consists of Gabor wavelet elements at multiple scales instead of a fixed scale. In addition to Gabor wavelet elements, we also include the center-surround difference of Gaussian wavelet elements in the dictionary. Such isotropic wavelet elements are of large scales, and they mainly capture the regional contrasts in the images. In the template in the last row, the number of selected wavelet elements is 50. Larger Gabor wavelet elements are illustrated by bars of lighter shades. The difference of Gaussian elements are illustrated by circles. The remaining images are reconstructed by such multi-scale wavelet representations, where each image is a linear superposition of the Gabor and difference of Gaussian wavelet elements of the corresponding deformed templates.

While selecting the wavelet elements of the active basis, we also estimate the distributions of their coefficients from the training images. This gives us a statistical model for the images. After learning this model, we can then use it to recognize the same type of objects in testing images. See Figure 4 for an example. The image on the left is the observed testing image. We scan the learned template of deer over this image, and at each location, we match the template to the image by deforming the learned template. The template matching is scored by the log-likelihood of the statistical model. We also scan the template over multiple resolutions of the image to account for the unknown scale of the object in the image. Then we choose the resolution and location of the image with the maximum likelihood score, and superpose on the image the deformed template matched to the image, as shown by the image on the right in Figure 4. This process can be accomplished by a cortex-like architecture of sum maps and max maps, to be described in Section 2.11. In machine learning and computer vision literature, detecting or classifying objects using the learned model is often called inference. The inference algorithm is often a part of the learning algorithm. For the active basis model, both learning and inference can be formulated as maximum likelihood estimation problems.

FIG. 4. *Left: Testing image. Right: Object is detected and sketched by the deformed template.*

2.1.4 Local maximum pooling and complex V1 cells

Besides wavelet sparse coding theory for V1 simple cells, another inspiration to the active basis model also comes from neuroscience. Riesenhuber and Poggio [17] observed that the complex cells of the primary visual cortex or V1 appear to perform local maximum pooling of the responses from simple cells. From the perspective of the active basis model, this corresponds to estimating the perturbations of the wavelet elements of the active basis template, so that the template is deformed to match the observed image. Therefore, if we are to believe Olshausen and Field's theory on wavelet sparse coding [15] and Riesenhuber and Poggio's theory on local maximum pooling, then the active basis model seems to be a very natural logical consequence.

In the following subsections we shall describe in detail wavelet sparse coding, the active basis model, and the learning and inference algorithms.

2.2 An Overcomplete Dictionary of Gabor Wavelets

The Gabor wavelets are translated, rotated and dilated versions of the following function:

$$G(x_1, x_2) \propto \exp\{ -[(x_1/\sigma_1)^2 + (x_2/\sigma_2)^2]/2 \} e^{ix_1},$$

which is sine-cosine wave multiplied by a Gaussian function. The Gaussian function is elongated along the x_2 -axis, with $\sigma_2 > \sigma_1$, and the sine-cosine wave propagates along the shorter x_1 -axis. We truncate the function to make it locally supported on a finite rectangular domain, so that it has a well defined length and width.

We then translate, rotate and dilate $G(x_1, x_2)$ to obtain a general form of the Gabor wavelets:

$$B_{x_1, x_2, s, \alpha}(x'_1, x'_2) = G(\tilde{x}_1/s, \tilde{x}_2/s)/s^2,$$

where

$$\begin{aligned} \tilde{x}_1 &= (x'_1 - x_1) \cos \alpha + (x'_2 - x_2) \sin \alpha, \\ \tilde{x}_2 &= -(x'_1 - x_1) \sin \alpha + (x'_2 - x_2) \cos \alpha. \end{aligned}$$

Writing $x = (x_1, x_2)$, each $B_{x, s, \alpha}$ is a localized function, where $x = (x_1, x_2)$ is the central location, s is the scale parameter, and α is the orientation. The frequency of the wave propagation in $B_{x, s, \alpha}$ is $\omega = 1/s$. $B_{x, s, \alpha} = (B_{x, s, \alpha, 0}, B_{x, s, \alpha, 1})$, where $B_{x, s, \alpha, 0}$ is the even-symmetric Gabor cosine component, and $B_{x, s, \alpha, 1}$ is the odd-symmetric Gabor sine component. We always use Gabor wavelets as pairs of cosine and sine components. We normalize both the

Gabor sine and cosine components to have zero mean and unit ℓ_2 norm. For each $B_{x, s, \alpha}$, the pair $B_{x, s, \alpha, 0}$ and $B_{x, s, \alpha, 1}$ are orthogonal to each other.

The dictionary of Gabor wavelets is

$$\Omega = \{B_{x, s, \alpha}, \forall (x, s, \alpha)\}.$$

We can discretize the orientation so that $\alpha \in \{o\pi/O, o = 0, \dots, O-1\}$, that is, O equally spaced orientations (the default value of O is 15 in our experiments). In this article we mostly learn the active basis template at a fixed scale s . The dictionary Ω is called "overcomplete" because the number of wavelet elements in Ω is larger than the number of pixels in the image domain, since at each pixel, there can be many wavelet elements tuned to different orientations and scales.

For an image $\mathbf{I}(x)$, with $x \in D$, where D is a set of pixels, such as a rectangular grid, we can project it onto a Gabor wavelet $B_{x, s, \alpha, \eta}$, $\eta = 0, 1$. The projection of \mathbf{I} onto $B_{x, s, \alpha, \eta}$, or the Gabor filter response at (x, s, α) , is

$$\langle \mathbf{I}, B_{x, s, \alpha, \eta} \rangle = \sum_{x'} \mathbf{I}(x') B_{x, s, \alpha, \eta}(x').$$

The summation is over the finite support of $B_{x, s, \alpha, \eta}$. We write $\langle \mathbf{I}, B_{x, s, \alpha} \rangle = (\langle \mathbf{I}, B_{x, s, \alpha, 0} \rangle, \langle \mathbf{I}, B_{x, s, \alpha, 1} \rangle)$. The local energy is

$$|\langle \mathbf{I}, B_{x, s, \alpha} \rangle|^2 = \langle \mathbf{I}, B_{x, s, \alpha, 0} \rangle^2 + \langle \mathbf{I}, B_{x, s, \alpha, 1} \rangle^2.$$

$|\langle \mathbf{I}, B_{x, s, \alpha} \rangle|^2$ is the local spectrum or the magnitude of the local wave in image \mathbf{I} at (x, s, α) .

Let

$$\sigma_s^2 = \frac{1}{|D|O} \sum_{\alpha} \sum_{x \in D} |\langle \mathbf{I}, B_{x, s, \alpha} \rangle|^2,$$

where $|D|$ is the number of pixels in \mathbf{I} , and O is the total number of orientations. For each image \mathbf{I} , we normalize it to $\mathbf{I} \leftarrow \mathbf{I}/\sigma_s$, so that different images are comparable.

2.3 Matching Pursuit Algorithm

For an image $\mathbf{I}(x)$ where $x \in D$, we seek to represent it by

$$(1) \quad \mathbf{I} = \sum_{i=1}^n c_i B_{x_i, s, \alpha_i} + U,$$

where $(B_{x_i, s, \alpha_i}, i = 1, \dots, n) \subset \Omega$ is a set of Gabor wavelet elements selected from the dictionary Ω , c_i is the coefficient, and U is the unexplained residual image. Recall that each B_{x_i, s, α_i} is a pair of

Gabor cosine and sine components. So $B_{x_i,s,\alpha_i} = (B_{x_i,s,\alpha_i,0}, B_{x_i,s,\alpha_i,1})$, $c_i = (c_{i,0}, c_{i,1})$, and $c_i B_{x_i,s,\alpha_i} = c_{i,0} B_{x_i,s,\alpha_i,0} + c_{i,1} B_{x_i,s,\alpha_i,1}$. We fix the scale parameter s .

In the representation (1), n is often assumed to be small, for example, $n = 50$. So the representation (1) is called sparse representation or sparse coding. This representation translates a raw intensity image with a huge number of pixels into a sketch with only a small number of strokes represented by $\mathbf{B} = (B_{x_i,s,\alpha_i}, i = 1, \dots, n)$. Because of the sparsity, \mathbf{B} captures the most visually meaningful elements in the image. The set of wavelet elements $\mathbf{B} = (B_{x_i,s,\alpha_i}, i = 1, \dots, n)$ can be selected from Ω by the matching pursuit algorithm [13], which seeks to minimize $\|\mathbf{I} - \sum_{i=1}^n c_i B_{x_i,s,\alpha_i}\|^2$ by a greedy scheme.

ALGORITHM 0 (Matching pursuit algorithm).

0. Initialize $i \leftarrow 0$, $U \leftarrow \mathbf{I}$.
1. Let $i \leftarrow i + 1$. Let $(x_i, \alpha_i) = \arg \max_{x,\alpha} |\langle U, B_{x,s,\alpha} \rangle|^2$.
2. Let $c_i = \langle U, B_{x_i,s,\alpha_i} \rangle$. Update $U \leftarrow U - c_i \times B_{x_i,s,\alpha_i}$.
3. Stop if $i = n$, else go back to 1.

In the above algorithm, it is possible that a wavelet element is selected more than once, but this is extremely rare for real images. As to the choice of n or the stopping criterion, we can stop the algorithm if $|c_i|$ is below a threshold.

Readers who are familiar with the so-called “large p and small n ” problem in linear regression may have recognized that wavelet sparse coding is a special case of this problem, where \mathbf{I} is the response vector, and each $B_{x,s,\alpha} \in \Omega$ is a predictor vector. The matching pursuit algorithm is actually the forward selection procedure for variable selection.

The forward selection algorithm in general can be too greedy. But for image representation, each Gabor wavelet element only explains away a small part of the image data, and we usually pursue the elements at a fixed scale, so such a forward selection procedure is not very greedy in this context.

2.4 Matching Pursuit for Multiple Images

Let $\{\mathbf{I}_m, m = 1, \dots, M\}$ be a set of training images defined on a common rectangle lattice D , and let us suppose that these images come from the same object category, where the objects appear at the same pose, location and scale in these images. We

can model these images by a common set of Gabor wavelet elements,

$$(2) \quad \mathbf{I}_m = \sum_{i=1}^n c_{m,i} B_{x_i,s,\alpha_i} + U_m, \quad m = 1, \dots, M.$$

$\mathbf{B} = (B_{x_i,s,\alpha_i}, i = 1, \dots, n)$ can be considered a common template for these training images. Model (2) is an extension of model (1).

We can select these elements by applying the matching pursuit algorithm on these multiple images simultaneously. The algorithm seeks to minimize $\sum_{m=1}^M \|\mathbf{I}_m - \sum_{i=1}^n c_{m,i} B_{x_i,s,\alpha_i}\|^2$ by a greedy scheme.

ALGORITHM 1 (Matching pursuit on multiple images).

0. Initialize $i \leftarrow 0$. For $m = 1, \dots, M$, initialize $U_m \leftarrow \mathbf{I}_m$.
1. $i \leftarrow i + 1$. Select

$$(x_i, \alpha_i) = \arg \max_{x,\alpha} \sum_{m=1}^M |\langle U_m, B_{x,s,\alpha} \rangle|^2.$$

2. For $m = 1, \dots, M$, let $c_{m,i} = \langle U_m, B_{x_i,s,\alpha_i} \rangle$, and update $U_m \leftarrow U_m - c_{m,i} B_{x_i,s,\alpha_i}$.
3. Stop if $i = n$, else go back to 1.

Algorithm 1 is similar to Algorithm 0. The difference is that, in Step 1, (x_i, α_i) is selected by maximizing the sum of the squared responses.

2.5 Active Basis and Local Maximum Pooling

The objects in the training images share similar shapes, but there can still be considerable variations in their shapes. In order to account for the shape deformations, we introduce the perturbations to the common template, and the model becomes

$$(3) \quad \mathbf{I}_m = \sum_{i=1}^n c_{m,i} B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}} + U_m, \quad m = 1, \dots, M.$$

Again, $\mathbf{B} = (B_{x_i,s,\alpha_i}, i = 1, \dots, n)$ can be considered a common template for the training images, but this time, this template is deformable. Specifically, for each image \mathbf{I}_m , the wavelet element B_{x_i,s,α_i} is perturbed to $B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}}$, where $\Delta x_{m,i}$ is the perturbation in location, and $\Delta \alpha_{m,i}$ is the perturbation in orientation. $\mathbf{B}_m = (B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}}, i = 1, \dots, n)$ can be considered the deformed template for coding image \mathbf{I}_m . We call the basis formed by

$\mathbf{B} = (B_{x_i, s, \alpha_i}, i = 1, \dots, n)$ the active basis, and we call $(\Delta x_{m,i}, \Delta \alpha_{m,i}, i = 1, \dots, n)$ the activities or perturbations of the basis elements for image m . Model (3) is an extension of model (2).

Figure 2 illustrates three examples of active basis templates. In the second and third rows the templates in the first column are $\mathbf{B} = (B_{x_i, s, \alpha_i}, i = 1, \dots, n)$. The scale parameter s in the second row is smaller than the s in the third row. For each row, the templates in the remain columns are the deformed templates $\mathbf{B}_m = (B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}}, i = 1, \dots, n)$, for $m = 1, \dots, 8$. The template in the last row should be more precisely represented by $\mathbf{B} = (B_{x_i, s_i, \alpha_i}, i = 1, \dots, n)$, where each element has its own s_i automatically selected together with (x_i, α_i) . In this article we focus on the situation where we fix s (default length of the wavelet element is 17 pixels).

For the activity or perturbation of a wavelet element $B_{x, s, \alpha}$, we assume that $\Delta x = (d \cos \alpha, d \sin \alpha)$, with $d \in [-b_1, b_1]$. That is, we allow $B_{x, s, \alpha}$ to shift its location along its normal direction. We also assume $\Delta \alpha \in [-b_2, b_2]$. b_1 and b_2 are the bounds for the allowed displacements in location and orientation (default values: $b_1 = 6$ pixels, and $b_2 = \pi/15$). We define

$$A(\alpha) = \{(\Delta x = (d \cos \alpha, d \sin \alpha), \Delta \alpha) : d \in [-b_1, b_1], \Delta \alpha \in [-b_2, b_2]\}$$

the set of all possible activities for a basis element tuned to orientation α .

We can continue to apply the matching pursuit algorithm to the multiple training images, the only difference is that we add a local maximum pooling operation in Steps 1 and 2. The following algorithm is a greedy procedure to minimize the least squares criterion:

$$(4) \quad \sum_{m=1}^M \left\| \mathbf{I}_m - \sum_{i=1}^n c_{m,i} B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}} \right\|^2.$$

ALGORITHM 2 (Matching pursuit with local maximum pooling).

0. Initialize $i \leftarrow 0$. For $m = 1, \dots, M$, initialize $U_m \leftarrow \mathbf{I}_m$.

1. $i \leftarrow i + 1$. Select

$$(x_i, \alpha_i)$$

$$= \arg \max_{x, \alpha} \sum_{m=1}^M \max_{(\Delta x, \Delta \alpha) \in A(\alpha)} |\langle U_m, B_{x + \Delta x, s, \alpha + \Delta \alpha} \rangle|^2.$$

2. For $m = 1, \dots, M$, retrieve

$$(\Delta x_{m,i}, \Delta \alpha_{m,i}) = \arg \max_{(\Delta x, \Delta \alpha) \in A(\alpha_i)} |\langle U_m, B_{x_i + \Delta x, s, \alpha_i + \Delta \alpha} \rangle|^2.$$

Let $c_{m,i} \leftarrow \langle U_m, B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}} \rangle$, and update $U_m \leftarrow U_m - c_{m,i} B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}}$.

3. Stop if $i = n$, else go back to 1.

Algorithm 2 is similar to Algorithm 1. The difference is that we add an extra local maximization operation in Step 1: $\max_{(\Delta x, \Delta \alpha) \in A(\alpha)} |\langle U_m, B_{x + \Delta x, s, \alpha + \Delta \alpha} \rangle|^2$. With (x_i, α_i) selected in Step 1, Step 2 retrieves the corresponding maximal $(\Delta x, \Delta \alpha)$ for each image.

We can rewrite Algorithm 2 by defining $R_m(x, \alpha) = \langle U_m, B_{x, s, \alpha} \rangle$. Then instead of updating the residual image U_m in Step 2, we can update the responses $R_m(x, \alpha)$.

ALGORITHM 2.1 (Matching pursuit with local maximum pooling).

0. Initialize $i \leftarrow 0$. For $m = 1, \dots, M$, initialize $R_m(x, \alpha) \leftarrow \langle \mathbf{I}_m, B_{x, s, \alpha} \rangle$ for all (x, α) .

1. $i \leftarrow i + 1$. Select

$$(x_i, \alpha_i)$$

$$= \arg \max_{x, \alpha} \sum_{m=1}^M \max_{(\Delta x, \Delta \alpha) \in A(\alpha)} |R_m(x + \Delta x, \alpha + \Delta \alpha)|^2.$$

2. For $m = 1, \dots, M$, retrieve

$$(\Delta x_{m,i}, \Delta \alpha_{m,i}) = \arg \max_{(\Delta x, \Delta \alpha) \in A(\alpha_i)} |R_m(x_i + \Delta x, \alpha_i + \Delta \alpha)|^2.$$

Let $c_{m,i} \leftarrow R_m(x_i + \Delta x_{m,i}, \alpha_i + \Delta \alpha_{m,i})$, and update

$$R_m(x, \alpha)$$

$$\leftarrow R_m(x, \alpha) - c_{m,i} \langle B_{x, s, \alpha}, B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}} \rangle.$$

3. Stop if $i = n$, else go back to 1.

2.6 Shared Sketch Algorithm

Finally, we come to the shared sketch algorithm that we actually used in the experiments in this paper. The algorithm involves two modifications to Algorithm 2.1.

ALGORITHM 3 (Shared sketch algorithm).

0. Initialize $i \leftarrow 0$. For $m = 1, \dots, M$, initialize $R_m(x, \alpha) \leftarrow \langle \mathbf{I}_m, B_{x,s,\alpha} \rangle$ for all (x, α) .

1. $i \leftarrow i + 1$. Select

$$(x_i, \alpha_i) = \arg \max_{x, \alpha} \sum_{m=1}^M \max_{(\Delta x, \Delta \alpha) \in A(\alpha)} h(|R_m(x + \Delta x, \alpha + \Delta \alpha)|^2).$$

2. For $m = 1, \dots, M$, retrieve

$$(\Delta x_{m,i}, \Delta \alpha_{m,i}) = \arg \max_{(\Delta x, \Delta \alpha) \in A(\alpha_i)} |R_m(x_i + \Delta x, \alpha_i + \Delta \alpha)|^2.$$

Let $c_{m,i} \leftarrow R_m(x_i + \Delta x_{m,i}, \alpha_i + \Delta \alpha_{m,i})$, and update $R_m(x, \alpha) \leftarrow 0$ if

$$\text{corr}(B_{x,s,\alpha}, B_{x_i+\Delta x_{m,i},s,\alpha_i+\Delta \alpha_{m,i}}) > 0.$$

3. Stop if $i = n$, else go back to 1.

The two modifications are as follows:

(1) In Step 1, we change $|R_m(x + \Delta x, \alpha + \Delta \alpha)|^2$ to $h(|R_m(x + \Delta x, \alpha + \Delta \alpha)|^2)$ where $h(\cdot)$ is a sigmoid function, which increases from 0 to a saturation level ξ (default: $\xi = 6$),

$$(5) \quad h(r) = \xi \left[\frac{2}{1 + e^{-2r/\xi}} - 1 \right].$$

Intuitively, $\sum_{m=1}^M \max_{(\Delta x, \Delta \alpha) \in A(\alpha)} h(|R_m(x + \Delta x, \alpha + \Delta \alpha)|^2)$ can be considered the sum of the votes from all the images for the location and orientation (x, α) , where each image contributes $\max_{(\Delta x, \Delta \alpha) \in A(\alpha)} h(|R_m(x + \Delta x, \alpha + \Delta \alpha)|^2)$. The sigmoid transformation prevents a small number of images from contributing very large values. As a result, the selection of (x, α) is a more “democratic” choice than in Algorithm 2, and the selected element seeks to sketch as many edges in the training images as possible. In the next section we shall formally justify the use of sigmoid transformation by a statistical model.

(2) In Step 2, we update $R_m(x, \alpha) \leftarrow 0$ if $B_{x,s,\alpha}$ is not orthogonal to $B_{x_i+\Delta x_{m,i},s,\alpha_i+\Delta \alpha_{m,i}}$. That is, we enforce the orthogonality of the basis $\mathbf{B}_m = (B_{x_i+\Delta x_{m,i},s,\alpha_i+\Delta \alpha_{m,i}}, i = 1, \dots, n)$ for each training image m . Our experience with matching pursuit is that it usually selects elements that have little overlap with each other. So for computational convenience, we simply enforce that the selected elements

are orthogonal to each other. For two Gabor wavelets B_1 and B_2 , we define their correlation as $\text{corr}(B_1, B_2) = \sum_{\eta_1=0}^1 \sum_{\eta_2=0}^1 \langle B_{1,\eta_1}, B_{2,\eta_2} \rangle^2$, that is, the sum of squared inner products between the sine and cosine components of B_1 and B_2 . In practical implementation, we allow small correlations between selected elements, that is, we update $R_m(x, \alpha) \leftarrow 0$ if $\text{corr}(B_{x,s,\alpha}, B_{x_i+\Delta x_{m,i},s,\alpha_i+\Delta \alpha_{m,i}}) > \varepsilon$ (the default value of $\varepsilon = 0.1$).

2.7 Statistical Modeling of Images

In this subsection we develop a statistical model for \mathbf{I}_m . A statistical model is not only important for justifying Algorithm 3 for learning the active basis template, it also enables us to use the learned template to recognize the objects in testing images, because we can use the log-likelihood to score the matching between the learned template and the image data.

The statistical model is based on the decomposition $\mathbf{I}_m = \sum_{i=1}^m c_{m,i} B_{x_i+\Delta x_{m,i},s,\alpha_i+\Delta \alpha_{m,i}} + U_m$, where $\mathbf{B}_m = (B_{x_i+\Delta x_{m,i},s,\alpha_i+\Delta \alpha_{m,i}}, i = 1, \dots, n)$ is orthogonal, and $c_{m,i} = \langle \mathbf{I}_m, B_{x_i+\Delta x_{m,i},s,\alpha_i+\Delta \alpha_{m,i}} \rangle$, so U_m lives in the subspace that is orthogonal to \mathbf{B}_m . In order to specify a statistical model for \mathbf{I}_m given \mathbf{B}_m , we only need to specify the distribution of $(c_{m,i}, i = 1, \dots, n)$ and the conditional distribution of U_m given $(c_{m,i}, i = 1, \dots, n)$.

The least squares criterion (4) that drives Algorithm 2 implicitly assumes that U_m is white noise, and $c_{m,i}$ follows a flat prior distribution. These assumptions are wrong. There can be occasional strong edges in the background, but a white noise U_m cannot account for strong edges. The distribution of $c_{m,i}$ should be estimated from the training images, instead of being assumed to be a flat distribution.

In this work we choose to estimate the distribution of $c_{m,i}$ from the training images by fitting an exponential family model to the sample $\{c_{m,i}, m = 1, \dots, M\}$ obtained from the training images, and we assume that the conditional distribution of U_m given $(c_{m,i}, i = 1, \dots, n)$ is the same as the corresponding conditional distribution in the natural images. Such a conditional distribution can account for occasional strong edges in the background, and it is the use of such a conditional distribution of U_m as well as the exponential family model for $c_{m,i}$ that leads to the sigmoid transformation in Algorithm 3. Intuitively, a large response $|R_m(x + \Delta x, \alpha + \Delta \alpha)|^2$ indicates that there can be an edge at $(x + \Delta x, \alpha + \Delta \alpha)$. Because

an edge can also be accounted for by the distribution of U_m in the natural images, a large response should not be taken at its face value for selecting the basis elements. Instead, it should be discounted by a transformation such as $h(\cdot)$ in Algorithm 3.

2.8 Density Substitution and Projection Pursuit

More specifically, we adopt the density substitution scheme of projection pursuit [10] to construct a statistical model. We start from a reference distribution $q(\mathbf{I})$. In this article we assume that $q(\mathbf{I})$ is the distribution of all the natural images. We do not need to know $q(\mathbf{I})$ explicitly beyond the marginal distribution $q(c)$ of $c = \langle \mathbf{I}, B_{x,s,\alpha} \rangle$ under $q(\mathbf{I})$. Because $q(\mathbf{I})$ is stationary and isotropic, $q(c)$ is the same for different (x, α) . $q(c)$ is a heavy tailed distribution because there are edges in the natural images. $q(c)$ can be estimated from the natural images by pooling a histogram of $\{\langle \mathbf{I}, B_{x,s,\alpha} \rangle, \forall \mathbf{I}, \forall (x, \alpha)\}$ where $\{\mathbf{I}\}$ is a sample of the natural images.

Given $\mathbf{B}_m = (B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}}, i = 1, \dots, n)$, we modify the reference distribution $q(\mathbf{I}_m)$ to a new distribution $p(\mathbf{I}_m)$ by changing the distributions of $c_{m,i}$. Let $p_i(c)$ be the distribution of $c_{m,i}$ pooled from $\{c_{m,i}, m = 1, \dots, M\}$, which are obtained from the training images $\{\mathbf{I}_m, m = 1, \dots, M\}$. Then we change the distribution of $c_{m,i}$ from $q(c)$ to $p_i(c)$, for each $i = 1, \dots, n$, while keeping the conditional distribution of U_m given $(c_{m,i}, i = 1, \dots, n)$ unchanged. This leads us to

$$(6) \quad \begin{aligned} p(\mathbf{I}_m | \mathbf{B}_m = (B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}}, i = 1, \dots, n)) \\ = q(\mathbf{I}_m) \prod_{i=1}^n \frac{p_i(c_{m,i})}{q(c_{m,i})}, \end{aligned}$$

where we assume that $(c_{m,i}, i = 1, \dots, n)$ are independent under both $q(\mathbf{I}_m)$ and $p(\mathbf{I}_m | \mathbf{B}_m)$, for orthogonal \mathbf{B}_m . The conditional distributions of U_m given $(c_{m,i}, i = 1, \dots, n)$ under $p(\mathbf{I}_m | \mathbf{B}_m)$ and $q(\mathbf{I}_m)$ are canceled out in $p(\mathbf{I}_m | \mathbf{B}_m) / q(\mathbf{I}_m)$ because they are the same. The Jacobians are also the same and are canceled out. So $p(\mathbf{I}_m | \mathbf{B}_m) / q(\mathbf{I}_m) = \prod_{i=1}^n p_i(c_{m,i}) / q(c_{m,i})$.

The following are three perspectives to view model (6):

(1) Classification: we may consider $q(\mathbf{I})$ as representing the negative examples, and $\{\mathbf{I}_m\}$ are positive examples. We want to find the basis elements $(B_{x_i, s, \alpha_i}, i = 1, \dots, n)$ so that the projections $c_{m,i} = \langle \mathbf{I}_m, B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}} \rangle$ for $i = 1, \dots, n$ distinguish the positive examples from the negative examples.

(2) Hypothesis testing: we may consider $q(\mathbf{I})$ as representing the null hypothesis, and the observed histograms of $c_{m,i}, i = 1, \dots, n$ are the test statistics that are used to reject the null hypothesis.

(3) Coding: we choose to code $c_{m,i}$ by $p_i(c)$ instead of $q(c)$, while continuing to code U_m by the conditional distribution of U_m given $(c_{m,i}, i = 1, \dots, n)$ under $q(\mathbf{I})$.

For all the three perspectives, we need to choose B_{x_i, s, α_i} so that there is big contrast between $p_i(c)$ and $q(c)$. The shared sketch process can be considered as sequentially flipping dimensions of $q(\mathbf{I}_m)$ from $q(c)$ to $p_i(c)$ to fit the observed images. It is essentially a projection pursuit procedure, with an additional local maximization step for estimating the activities of the basis elements.

2.9 Exponential Tilting and Saturation Transformation

While $p_i(c)$ can be estimated from $\{c_{m,i}, m = 1, \dots, M\}$ by pooling a histogram, we choose to parametrize $p_i(c)$ with a single parameter so that it can be estimated from even a single image.

We assume $p_i(c)$ to be the following exponential family model:

$$(7) \quad p(c; \lambda) = \frac{1}{Z(\lambda)} \exp\{\lambda h(r)\} q(c),$$

where $\lambda > 0$ is the parameter. For $c = (c_0, c_1)$, $r = |c|^2 = c_0^2 + c_1^2$,

$$Z(\lambda) = \int \exp\{\lambda h(r)\} q(c) dc = \mathbb{E}_q[\exp\{\lambda h(r)\}]$$

is the normalizing constant. $h(r)$ is a monotone increasing function. We assume $p_i(c) = p(c; \lambda_i)$, which accounts for the fact that the squared responses $\{|c_{m,i}|^2 = |\langle \mathbf{I}_m, B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}} \rangle|^2, m = 1, \dots, M\}$ in the positive examples are in general larger than those in the natural images, because $B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}}$ tends to sketch a local edge segment in each \mathbf{I}_m . As mentioned before, $q(c)$ is estimated by pooling a histogram from the natural images.

We argue that $h(r)$ should be a saturation transformation in the sense that as $r \rightarrow \infty$, $h(r)$ approaches a finite number. The sigmoid transformation in (5) is such a transformation. The reason for such a transformation is as follows. Let $q(r)$ be the distribution of $r = |c|^2 = |\langle \mathbf{I}, B \rangle|^2$ under $q(c)$ where $\mathbf{I} \sim q(\mathbf{I})$. We may implicitly model $q(r)$ as a mixture of $p_{\text{on}}(r)$ and

$p_{\text{off}}(r)$, where p_{on} is the distribution of r when B is on an edge in \mathbf{I} , and p_{off} is the distribution of r when B is not on an edge in \mathbf{I} . $p_{\text{on}}(r)$ has a much heavier tail than $p_{\text{off}}(r)$. Let $q(r) = (1 - \rho_0)p_{\text{off}}(r) + \rho_0 p_{\text{on}}(r)$, where ρ_0 is the proportion of edges in the natural images. Similarly, let $p_i(r)$ be the distribution of $r = |c|^2$ under $p_i(c)$. We can model $p_i(r) = (1 - \rho_i)p_{\text{off}}(r) + \rho_i p_{\text{on}}(r)$, where $\rho_i > \rho_0$, that is, the proportion of edges sketched by the selected basis element is higher than the proportion of edges in the natural images. Then, as $r \rightarrow \infty$, $p_i(r)/q(r) \rightarrow \rho_i/\rho_0$, which is a constant. Therefore, $h(r)$ should saturate as $r \rightarrow \infty$.

2.10 Maximum Likelihood Learning and Pursuit Index

Now we can justify the shared sketch algorithm as a greedy scheme for maximizing the log-likelihood. With parametrization (7) for the statistical model (6), the log-likelihood is

$$(8) \quad \sum_{m=1}^M \sum_{i=1}^n \log \frac{p_i(c_{m,i})}{q(c_{m,i})} = \sum_{i=1}^n \left[\lambda_i \sum_{m=1}^M h(|\langle \mathbf{I}_m, B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}} \rangle|^2) - M \log Z(\lambda_i) \right].$$

We want to estimate the locations and orientations of the elements of the active basis, $(x_i, \alpha_i, i = 1, \dots, n)$, the activities of these elements, $(\Delta x_{m,i}, \Delta \alpha_{m,i}, i = 1, \dots, n)$, and the weights, $(\lambda_i, i = 1, \dots, n)$, by maximizing the log-likelihood (8), subject to the constraints that $\mathbf{B}_m = (B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}}, i = 1, \dots, n)$ is orthogonal for each m .

First, we consider the problem of estimating the weight λ_i given \mathbf{B}_m . To maximize the log-likelihood (8) over λ_i , we only need to maximize

$$l_i(\lambda_i) = \lambda_i \sum_{m=1}^M h(|\langle \mathbf{I}_m, B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}} \rangle|^2) - M \log Z(\lambda_i).$$

By setting $l'_i(\lambda_i) = 0$, we get the well-known form of the estimating equation for the exponential family model,

$$(9) \quad \mu(\lambda_i) = \frac{1}{M} \sum_{m=1}^M h(|\langle \mathbf{I}_m, B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}} \rangle|^2),$$

where the mean parameter $\mu(\lambda)$ of the exponential family model is

$$(10) \quad \begin{aligned} \mu(\lambda) &= E_\lambda[h(r)] \\ &= \frac{1}{Z(\lambda)} \int h(r) \exp\{\lambda h(r)\} q(r) dr. \end{aligned}$$

The estimating equation (9) can be solved easily because $\mu(\lambda)$ is a one-dimensional function. We can simply store this monotone function over a one-dimensional grid. Then we solve this equation by looking up the stored values, with the help of nearest neighbor linear interpolation for the values between the grid points. For each grid point of λ , $\mu(\lambda)$ can be computed by one-dimensional integration as in (10). Thanks to the independence assumption, we only need to deal with such one-dimensional functions, which relieves us from time consuming MCMC computations.

Next let us consider the problem of selecting (x_i, α_i) , and estimating the activity $(\Delta x_{m,i}, \Delta \alpha_{m,i})$ for each image \mathbf{I}_m . Let $\hat{\lambda}_i$ be the solution to the estimating equation (9). $l_i(\hat{\lambda}_i)$ is monotone in $\sum_{m=1}^M h(|\langle \mathbf{I}_m, B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}} \rangle|^2)$. Therefore, we need to find (x_i, α_i) , and $(\Delta x_{m,i}, \Delta \alpha_{m,i})$, by maximizing $\sum_{m=1}^M h(|\langle \mathbf{I}_m, B_{x_i + \Delta x_{m,i}, s, \alpha_i + \Delta \alpha_{m,i}} \rangle|^2)$. This justifies Step 1 of Algorithm 3, where $\sum_{m=1}^M h(|R_m(x + \Delta x, \alpha + \Delta \alpha)|^2)$ serves as the pursuit index.

2.11 SUM-MAX Maps for Template Matching

After learning the active basis model, in particular, the basis elements $\mathbf{B} = (B_{x_i, s, \alpha_i}, i = 1, \dots, n)$ and the weights $(\lambda_i, i = 1, \dots, n)$, we can use the learned model to find the object in a testing image \mathbf{I} , as illustrated by Figure 4. The testing image may not be defined on the same lattice as the training images. For example, the testing image may be larger than the training images. We assume that there is one object in the testing image, but we do not know the location of the object in the testing image. In order to detect the object, we scan the template over the testing image, and at each location x , we can deform the template and match it to the image patch around x . This gives us a log-likelihood score at each location x . Then we can find the maximum likelihood location \hat{x} that achieves the maximum of the log-likelihood score among all the x . After computing \hat{x} , we can then retrieve the activities of the elements of the active basis template centered at \hat{x} .

ALGORITHM 4 (Object detection by template matching).

1. For every x , compute

$$l(x) = \sum_{i=1}^n \left[\lambda_i \max_{(\Delta x, \Delta \alpha) \in A(\alpha_i)} h(|\langle \mathbf{I}, B_{x+x_i+\Delta x, s, \alpha_i+\Delta \alpha} \rangle|^2) - \log Z(\lambda_i) \right].$$

2. Select $\hat{x} = \arg \max_x l(x)$. For $i = 1, \dots, n$, retrieve

$$(\Delta x_i, \Delta \alpha_i) = \arg \max_{(\Delta x, \Delta \alpha) \in A(\alpha_i)} |\langle \mathbf{I}, B_{\hat{x}+x_i+\Delta x, s, \alpha_i+\Delta \alpha} \rangle|^2.$$

3. Return the location \hat{x} , and the deformed template $(B_{\hat{x}+x_i+\Delta x_i, s, \alpha_i+\Delta \alpha_i}, i = 1, \dots, n)$.

Figure 4 displays the deformed template $(B_{\hat{x}+x_i+\Delta x_i, s, \alpha_i+\Delta \alpha_i}, i = 1, \dots, n)$, which is superposed on the image on the right.

Step 1 of the above algorithm can be realized by a computational architecture called sum-max maps.

ALGORITHM 4.1 (sum-max maps).

1. For all (x, α) , compute $\text{SUM1}(x, \alpha) = h(|\langle \mathbf{I}, B_{x, s, \alpha} \rangle|^2)$.

2. For all (x, α) , compute

$$\text{MAX1}(x, \alpha) = \max_{(\Delta x, \Delta \alpha) \in A(\alpha)} \text{SUM1}(x + \Delta x, \alpha + \Delta \alpha).$$

3. For all x , compute $\text{SUM2}(x) = \sum_{i=1}^n [\lambda_i \times \text{MAX1}(x + x_i, \alpha_i) - \log Z(\lambda_i)]$.

$\text{SUM2}(x)$ is $l(x)$ in Algorithm 4.

The local maximization operation in Step 2 of Algorithm 4.1 has been hypothesized as the function of the complex cells of the primary visual cortex [17]. In the context of the active basis model, this operation can be justified as the maximum likelihood estimation of the activities. The shared sketch learning algorithm can also be written in terms of sum-max maps.

The activities $(\Delta x_{m,i}, \Delta \alpha_{m,i}, i = 1, \dots, n)$ should be treated as latent variables in the active basis model. However, in both learning and inference algorithms, we treat them as unknown parameters, and we maximize over them instead of integrating them out. According to Little and Rubin [12], maximizing the complete-data likelihood over the latent

variables may not lead to valid inference in general. However, in natural images, there is little noise, and the uncertainty in the activities is often very small. So maximizing over the latent variables can be considered a good approximation to integrating out the latent variables.

3. LEARNING ACTIVE BASIS TEMPLATES BY EM-TYPE ALGORITHMS

The shared sketch algorithm in the previous section requires that the objects in the training images $\{\mathbf{I}_m\}$ are of the same pose, at the same location and scale, and the lattice of \mathbf{I}_m is the bounding box of the object in \mathbf{I}_m . It is often the case that the objects may appear at different unknown locations, orientations and scales in $\{\mathbf{I}_m\}$. The unknown locations, orientations and scales can be incorporated into the image generation process as hidden variables. The template can still be learned by the maximum likelihood method.

3.1 Learning with Unknown Orientations

We start from a simple example of learning a horse template at the side view, where the horses can face either to the left or to the right. Figure 5 displays the results of EM learning. The three templates in the first row are the learned templates in the first three iterations of the EM algorithm. The rest of the figure displays the training images, and for each training image, a deformed template is displayed to the right of it. The EM algorithm correctly estimates the direction for each horse, as can be seen by how the algorithm flips the template to sketch each training image.

Let $\mathbf{B} = (B_i = B_{x_i, s, \alpha_i}, i = 1, \dots, n)$ be the deformable template of the horse, and $\mathbf{B}_m = (B_{m,i} = B_{x_i+\Delta x_{m,i}, s, \alpha_i+\Delta \alpha_{m,i}}, i = 1, \dots, n)$ be the deformed template for \mathbf{I}_m . Then \mathbf{I}_m can either be generated by \mathbf{B}_m or the mirror reflection of \mathbf{B}_m , that is, $(B_{R(x_i+\Delta x_{m,i}), s, -(\alpha_i+\Delta \alpha_{m,i})}, i = 1, \dots, n)$, where for $x = (x_1, x_2)$, $R(x) = (-x_1, x_2)$ (we assume that the template is centered at origin). We can introduce a hidden variable z_m to account for this uncertainty, so that $z_m = 1$ if \mathbf{I}_m is generated by \mathbf{B}_m , and $z_m = 0$ if \mathbf{I}_m is generated by the mirror reflection of \mathbf{B}_m . More formally, we can define $\mathbf{B}_m(z_m)$, so that $\mathbf{B}_m(1) = \mathbf{B}_m$, and $\mathbf{B}_m(0)$ is the mirror reflection of \mathbf{B}_m . Then we can assume the following mixture model: $z_m \sim \text{Bernoulli}(\rho)$, where ρ is the prior probability that $z_m = 1$, and $[\mathbf{I}_m | z_m] \sim p(\mathbf{I}_m | \mathbf{B}_m(z_m), \Lambda)$,

where $\Lambda = (\lambda_i, i = 1, \dots, n)$. We need to learn \mathbf{B} , and estimate Λ and ρ .

A simple observation is that $p(\mathbf{I}_m | \mathbf{B}_m(z_m)) = p(\mathbf{I}_m(z_m) | \mathbf{B}_m)$, where $\mathbf{I}_m(1) = \mathbf{I}_m$ and $\mathbf{I}_m(0)$ is the mirror reflection of \mathbf{I}_m . In other words, in the case of $z_m = 1$, we do not need to make any change to \mathbf{I}_m or \mathbf{B}_m . In the case of $z_m = 0$, we can either flip the template or flip the image, and these two alternatives will produce the same value for the likelihood function.

In the EM algorithm, the E-step imputes z_m for $m = 1, \dots, M$ using the current template \mathbf{B} . This means recognizing the orientation of the object in \mathbf{I}_m . Given z_m , we can change \mathbf{I}_m to $\mathbf{I}_m(z_m)$, so that

$\{\mathbf{I}_m(z_m)\}$ become aligned with each other, if z_m are imputed correctly. Then in the M-step, we can learn the template from the aligned images $\{\mathbf{I}_m(z_m)\}$ by the shared sketch algorithm.

The complete data log-likelihood for the m th observation is

$$\begin{aligned} \log p(\mathbf{I}_m, z_m | \mathbf{B}_m) \\ &= z_m \log p(\mathbf{I}_m | \mathbf{B}_m, \Lambda) \\ &\quad + (1 - z_m) \log p(\mathbf{I}_m(0) | \mathbf{B}_m, \Lambda) \\ &\quad + z_m \log \rho + (1 - z_m) \log(1 - \rho), \end{aligned}$$



FIG. 5. Template learned from images of horses facing two different directions. The first row displays the templates learned in the first 3 iterations of the EM algorithm. For each training image, the deformed template is plotted to the right of it. The number of training images is 57. The image size is 120×100 (width \times height). The number of elements is 40. The number of EM iterations is 3.

which is linear in z_m . So in the E-step, we only need to compute the predictive expectation of z_m ,

$$\begin{aligned}\hat{z}_m &= \Pr(z_m = 1 | \mathbf{B}_m, \Lambda, \rho) \\ &= \frac{\rho p(\mathbf{I}_m | \mathbf{B}_m, \Lambda)}{\rho p(\mathbf{I}_m | \mathbf{B}_m, \Lambda) + (1 - \rho) p(\mathbf{I}_m(0) | \mathbf{B}_m, \Lambda)}.\end{aligned}$$

Both $\log p(\mathbf{I}_m | \mathbf{B}_m, \Lambda)$ and $\log p(\mathbf{I}_m(0) | \mathbf{B}_m, \Lambda)$ are readily available in the M-step.

The M-step seeks to maximize the expectation of the complete-data log-likelihood,

$$\begin{aligned}(11) \quad & \sum_{i=1}^n \left[\lambda_i \sum_{m=1}^M (\hat{z}_m h(|\langle \mathbf{I}_m, B_{m,i} \rangle|^2) \right. \\ & \quad \left. + (1 - \hat{z}_m) h(|\langle \mathbf{I}_m(0), B_{m,i} \rangle|^2)) \right. \\ & \quad \left. - M \log Z(\lambda_i) \right] \\ (12) \quad & + \left[\log \rho \sum_{m=1}^M \hat{z}_m \right. \\ & \quad \left. + \log(1 - \rho) \left(M - \sum_{m=1}^M \hat{z}_m \right) \right].\end{aligned}$$

The maximization of (12) leads to $\hat{\rho} = \sum_{m=1}^M \hat{z}_m / M$. The maximization of (11) can be accomplished by the shared sketch algorithm, that is, Algorithm 3, with the following minor modifications:

(1) The training images become $\{\mathbf{I}_m, \mathbf{I}_m(0), m = 1, \dots, M\}$, that is, there are $2M$ training images instead of M images. Each \mathbf{I}_m contributes two copies, the original copy \mathbf{I}_m or $\mathbf{I}_m(1)$, and the mirror reflection $\mathbf{I}_m(0)$. This reflects the uncertainty in z_m . For each image \mathbf{I}_m , we attach a weight \hat{z}_m to \mathbf{I}_m , and a weight $1 - \hat{z}_m$ to $\mathbf{I}_m(0)$. Intuitively, a fraction of the horse in \mathbf{I}_m is at the same orientation as the current template, and a fraction of it is at the opposite orientation—a “Schrodinger horse” so to speak. We use $(\mathbf{J}_k, w_k, k = 1, \dots, 2M)$ to represent these $2M$ images and their weights.

(2) In Step 1 of the shared sketch algorithm, we select (x_i, α_i) by

$$\begin{aligned}(x_i, \alpha_i) &= \arg \max_{x, \alpha} \sum_{k=1}^{2M} w_k \max_{(\Delta x, \Delta \alpha) \in A(\alpha)} h(|R_k(x + \Delta x, \\ & \quad \alpha + \Delta \alpha)|^2).\end{aligned}$$

(3) The maximum likelihood estimating equation for λ_i is

$$\begin{aligned}\mu(\lambda_i) &= \frac{1}{M} \sum_{k=1}^{2M} w_k \max_{(\Delta x, \Delta \alpha) \in A(\alpha_i)} h(|R_k(x_i + \Delta x, \\ & \quad \alpha_i + \Delta \alpha)|^2),\end{aligned}$$

where the right-hand side is the weighted average obtained from the $2M$ training images.

(4) Along with the selection of B_{x_i, s, α_i} and the estimation of λ_i , we should calculate the template matching scores

$$\begin{aligned}\log p(\mathbf{J}_k | \mathbf{B}_k, \Lambda) &= \sum_{i=1}^n \left[\hat{\lambda}_i \max_{(\Delta x, \Delta \alpha) \in A(\alpha_i)} h(|R_k(x_i + \Delta x, \right. \\ & \quad \left. \alpha_i + \Delta \alpha)|^2) \right. \\ & \quad \left. - \log Z(\hat{\lambda}_i) \right],\end{aligned}$$

for $k = 1, \dots, 2M$. This gives us $\log p(\mathbf{I}_m | \mathbf{B}_m, \Lambda)$ and $\log p(\mathbf{I}_m(0) | \mathbf{B}_m, \Lambda)$, which can then be used in the E-step.

We initialize the algorithm by randomly generating $\hat{z}_m \sim \text{Unif}[0, 1]$, and then iterate between the M-step and the E-step. We stop the algorithm after a few iterations. Then we estimate $z_m = 1$ if $\hat{z}_m > 1/2$ and $z_m = 0$ otherwise.

In Figure 5 the results are obtained after 3 iterations of the EM algorithm. Initially, the learned template is quite symmetric, reflecting the confusion of the algorithm regarding the directions of the horses. Then the EM algorithm begins a process of “symmetry breaking” or “polarization.” The slight asymmetry in the initial template will push the algorithm toward favoring for each image the direction that is consistent with the majority direction. This process quickly leads to all the images aligned to one common direction.

Figure 6 shows another example where a template of a pigeon is learned from examples with mixed directions.

We can also learn a common template when the objects are at more than two different orientations in the training images. The algorithm is essentially the same as described above. Figure 7 displays the learning of the template of a baseball cap from examples where the caps turn to different orientations.



FIG. 6. Template learned from 11 images of pigeons facing different directions. The image size is 150×150 . The number of elements is 50. The number of iterations is 3.

The E-step involves rotating the images by matching to the current template, and the M-step learns the template from the rotated images.

3.2 Learning From Nonaligned Images

When the objects appear at different locations in the training images, we need to infer the unknown locations while learning the template. Figure 8 displays the template of a bike learned from the 7 training images where the objects appear at different locations and are not aligned. It also displays the deformed templates superposed on the objects in the training images.

In order to incorporate the unknown locations into the image generation process, let us assume that both the learned template $\mathbf{B} = (B_{x_i, s, \alpha_i}, i = 1, \dots, n)$ and the training images $\{\mathbf{I}_m\}$ are centered at origin. Then let us assume that the location of the object in image \mathbf{I}_m is $x^{(m)}$, which is assumed to be uniformly distributed within the image lattice of \mathbf{I}_m . Let us define $\mathbf{B}_m(x^{(m)}) = (B_{x^{(m)}+x_i+\Delta x_{m,i}, s, \alpha_i+\Delta \alpha_{m,i}}, i = 1, \dots, n)$ to be the deformed template obtained by translating the template \mathbf{B} from the origin to $x^{(m)}$ and then deforming it. Then the generative model for \mathbf{I}_m is $p(\mathbf{I}_m|\mathbf{B}_m(x^{(m)}), \Lambda)$.

Just like the example of learning the horse template, we can transfer the transformation of the template to the transformation of the image data, and

the latter transformation leads to the alignment of the images. Let us define $\mathbf{I}_m(x^{(m)})$ to be the image obtained by translating the image \mathbf{I}_m so that the center of $\mathbf{I}_m(x^{(m)})$ is $-x^{(m)}$. Then $p(\mathbf{I}_m|\mathbf{B}_m(x^{(m)}), \Lambda) = p(\mathbf{I}_m(x^{(m)})|\mathbf{B}_m, \Lambda)$. If we know $x^{(m)}$ for $m = 1, \dots, M$, then the images $\{\mathbf{I}_m(x^{(m)})\}$ are all aligned, so that we can learn a template from these aligned images by the shared sketch algorithm. On the other hand, if we know the template, we can use the template to recognize and locate the object in each \mathbf{I}_m by the inference algorithm, that is, Algorithm 4, using the sum-max maps, and identify $x^{(m)}$. Such considerations naturally lead to the iterative EM-type scheme.

The complete-data log-likelihood is

$$(13) \quad \sum_{i=1}^n \left[\lambda_i \sum_{m=1}^M h(|\langle \mathbf{I}_m(x^{(m)}), B_{x_i+\Delta x_{m,i}, s, \alpha_i+\Delta \alpha_{m,i}} \rangle|^2) - M \log Z(\lambda_i) \right].$$

In the E-step we perform the recognition task by calculating

$$p_m(x) = \Pr(x^{(m)} = x|\mathbf{B}, \Lambda) \propto p(\mathbf{I}_m(x)|\mathbf{B}_m, \Lambda), \quad \forall x.$$



FIG. 7. Template learned from 15 images of baseball caps facing different orientations. The image size is 100×100 . The number of elements is 40. The number of iterations is 5.

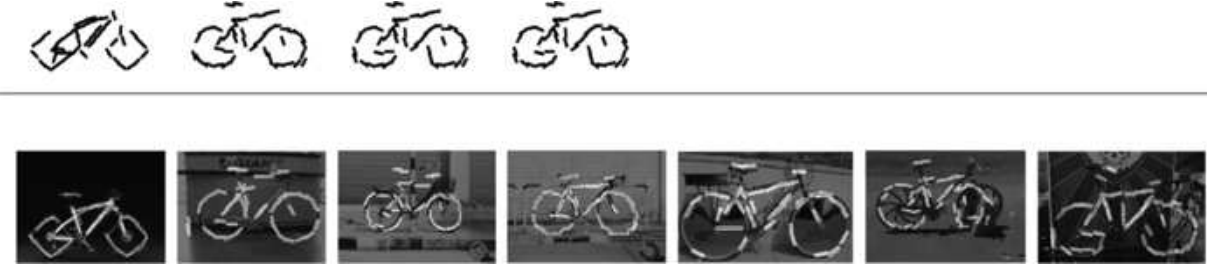


FIG. 8. The first row shows the sequence of templates learned in the first 3 iterations. The first one is the starting template, which is learned from the first training image. The second row shows the bikes detected by the learned template, where a deformed template is superposed on each training image. The size of the template is 225×169 . The number of elements is 50. The number of iterations is 3.

That is, we scan the template over the whole image \mathbf{I}_m , and at each location x , we evaluate the template matching between the image \mathbf{I}_m and the translated and deformed template $\mathbf{B}_m(x)$. $\log p(\mathbf{I}_m(x)|\mathbf{B}_m, \Lambda)$ is the SUM2(x) output by the sum-max maps in Algorithm 4.1. This gives us $p_m(x)$, which is the posterior or predictive distribution of the unknown location $x^{(m)}$ within the image lattice of \mathbf{I}_m . We can then use $p_m(x)$ to compute the expectation of the complete-data log-likelihood (13) in the E-step.

Our experience suggests that $p_m(x)$ is always highly peaked at a particular position. So instead of computing the average of (13), we simply impute $x^{(m)} = \arg \max_x p_m(x)$.

Then in the M-step, we maximize the complete data log-likelihood (13) by the shared sketch algorithm, that is, we learn the template \mathbf{B} from $\{\mathbf{I}_m(x^{(m)})\}$. This step performs supervised learning from the aligned images.

In our current experiment we initialize the algorithm by learning (\mathbf{B}, Λ) from the first image. In learning from this single image, we set $b_1 = b_2 = 0$, that is, we do not allow the elements $(B_{x_i, s, \alpha_i}, i = 1, \dots, n)$ to perturb. After that, we reset b_1 and b_2 to their default values, and iterate the recognition step and the supervised learning step.

In addition to the unknown locations, we also allow the uncertainty in scales. In the recognition step, for each \mathbf{I}_m , we search over a number of different resolutions of \mathbf{I}_m . We take $\mathbf{I}_m(x^{(m)})$ to be the optimal resolution that contains the maximum template matching score across all the resolutions.

In Figure 8 the first row displays the templates learned over the EM iterations. The first template is learned from the first training image. Figures 9–12 display more examples.

4. DISCUSSION

This paper experiments with EM-type algorithms for learning active basis models from training images where the objects may appear at unknown locations, orientations and scales. For more details on implementing the shared sketch algorithm, the reader is referred to [23] and the source code posted on the reproducibility page.

We would like to emphasize two aspects of the algorithms that are different from the usual EM algorithm. The first aspect is that the M-step involves the selection of the basis elements, in addition to the estimation of the associated parameters. The second aspect is that the performance of the algorithms can rely heavily on the initializations. In learning from nonaligned images, the algorithm is initialized by training the active basis model on a single image. Because of the simplicity of the model, it is possible to learn the model from a single image. In addition, the learning algorithm seems to converge within a few iterations.

4.1 Limitations

The active basis model is a simple extension of the wavelet representation. It is still very limited in the following aspects. The model cannot account for large deformations, articulate shapes, big changes in poses and view points, and occlusions. The current form of the model does not describe textures and lighting variations either. The current version of the learning algorithm only deal with situations where there is one object in each image. Also, we have tuned two parameters in our implementation. One is the image resize factor that we apply to the training images before the model is learned. Of course, for each experiment, a single resize factor is applied to all the training images. The other parameter is the number of elements in the active basis.

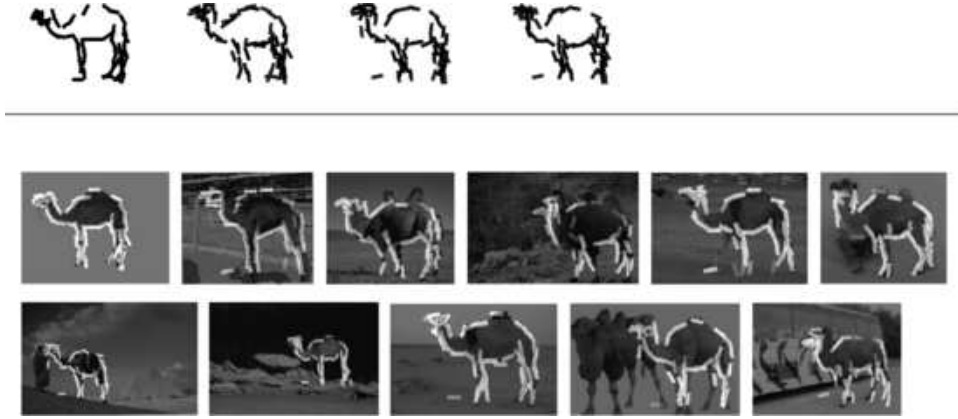


FIG. 9. The first row shows the sequence of templates learned in iterations 0, 1, 3, 5. The second and third rows show the camel images with superposed deformed templates. The size of the template is 192×145 . The number of elements is 60. The number of iterations is 5.

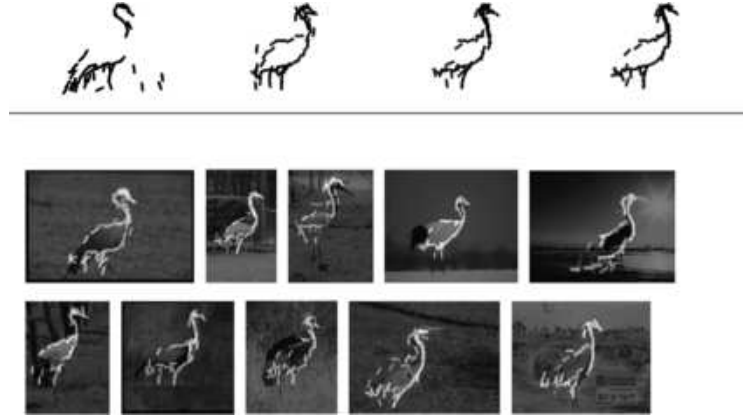


FIG. 10. The first row shows the sequence of templates learned in iterations 0, 1, 3, 5. The other rows show the crane images with superposed deformed templates. The size of the template is 285×190 . The number of elements is 50. The number of iterations is 5.

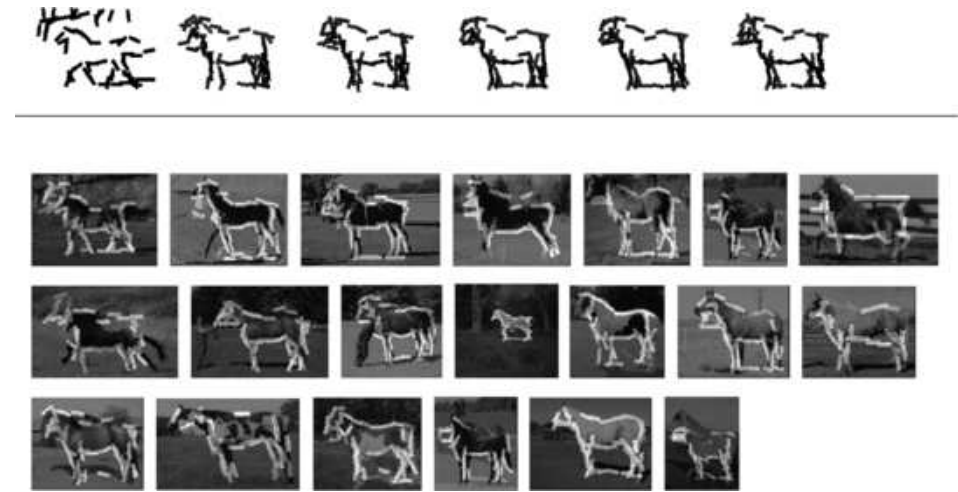


FIG. 11. The first row shows the sequence of templates learned in iterations 0, 2, 4, 6, 8, 10. The other rows show the horse images with superposed deformed templates. We use the first 20 images of the Weizmann horse data set [3], which are resized to half the original sizes. The size of the template is 158×116 . The number of elements is 60. The number of iterations is 10. The detection results on the rest of the images in this data set can be found in the reproducibility page.



FIG. 12. The size of the template is 240×180 . The number of elements is 60. The number of iterations is 3.

4.2 Possible Extensions

It is possible to extend the active basis model to address some of the above limitations. We shall discuss two directions of extensions. One is to use active basis models as parts of the objects. The other is to train active basis models by local learning.

Active basis models as part-templates: The active basis model is a composition of a number of Gabor wavelet elements. We can further compose multiple active basis models to represent more articulate shapes or to account for large deformations by allowing these active basis models to change their overall locations, scales and orientations within limited ranges. These active basis models serve as part-templates of the whole composite template. This is essentially a hierarchical recursive compositional structure [11, 25]. The inference or template matching can be based on a recursive structure of sum-max maps. Learning such a structure should be possible by extending the learning algorithms studied in this article. See [23] for preliminary results. See also [18, 20] for recent work on part-based models.

Local learning of multiple prototype templates: In each experiment we assume that all the training images share a common template. In reality, the training images may contain different types of objects, or different poses of the same type of objects. It is therefore necessary to learn multiple prototype templates. It is possible to do so by modifying the current learning algorithm. After initializing the algorithm by single image training, in the M-step, we can relearn the template only from the K images with the highest template matching scores, that is, we relearn the template from the K nearest neighbors of the current template. Such a scheme is consistent with the EM-clustering algorithm for fitting

mixture models. We can start the algorithm from every training image, so that we learn a local prototype template around each training image. Then we can trim and merge these prototypes. See [23] for preliminary results.

REPRODUCIBILITY

All the experimental results reported in this paper can be reproduced by the code that we have posted at <http://www.stat.ucla.edu/~ywu/ActiveBasis.html>.

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